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THE LONGITUDINAL VIBRATIONS OF AN ELASTIC CYLINDER UNDER LARGE AXIAL AND NORMAL TRACTIONS

A. J. WILLSON and P. J. MYERS†
Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, U.K.

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Abstract—The dispersion equation governing small-amplitude longitudinal vibrations, in their lowest mode, of an elastic cylinder upon which there are imposed large steady axial and normal tractions, is given in an approximate form valid for wavelengths large compared to the cylinder radius. The displacement and stress fields are also obtained. Three cases are considered in which the isotropic hyperelastic material of the cylinder is (i) incompressible, (ii) compressible and free from internal constraint, (iii) compressible but subject to Bell's constraint. The dispersion equation is illustrated graphically for certain specific strain-energy density functions, including those for a generalized Blatz–Ko material, a Mooney–Rivlin material and a model for biological tissue.

1. INTRODUCTION

In the course of their classical investigations into the vibrations of an elastic cylinder of circular cross-section Pochhammer (1876) and later, but independently, Chree (1886) [see also Love (1944)], obtained the approximate dispersion equation

$$\frac{\rho\omega^2}{k^2} = E\left[1 - \frac{\sigma^2(ka)^2}{2} + O(ka)^4\right], \tag{1}$$

governing the lowest mode of longitudinal vibration. Here ω, k are the angular frequency and wave-number, and E, σ, ρ, a the Young's modulus, Poisson's ratio, density and radius of the cylinder.

Our aim in this paper is to employ the methods of modern non-linear elasticity theory in order to derive the corresponding dispersion relation for small-amplitude waves in a cylinder subjected to large steady axial and normal tractions. The result may find application in the stability theory of cylinders under stress and in the propagation of ultrasonic waves along stressed bars. It turns out that for any homogeneous isotropic hyperelastic material it is possible to define quantities E^*, σ^* in terms of the strain-energy density function and the imposed tractions so that the derived dispersion equation is of broadly similar, but not identical, form to (1). Further, these quantities E^*, σ^* are simply related to the behaviour of the cylinder under incremental uniaxial static tension (or compression). This makes for ease of calculation on the one hand and of experimental verification on the other. Three cases are examined: first, the case in which the elastic material is compressible and free from any internal constraint, secondly the case of incompressible material and finally the very interesting situation in which the material is subject to the Bell constraint [see Bell (1983)]. The analysis in the last case may have important implications for the behaviour of the cylinder material as it approaches the plastic regime. Corresponding to these three cases, the results are illustrated for material obeying a particular generalization of the Blatz–Ko strain-energy function whose properties are discussed by the authors elsewhere [see Willson and Myers (1988)], for biological tissue [see Fung (1967)] and for Mooney–Rivlin material, where we recover a result derived by Suhubi (1965) [see also Eringen and Suhubi (1974)], and finally for material with a strain-energy function proposed by Ericksen to account for the experimental results of Bell (1983).

† Present address: GPT Limited, New Century Park, PO Box 53, Coventry CV3 1HJ, U.K.

2. COMPRESSIBLE MATERIAL

For a full account of the theory of non-linear elasticity the reader is referred to Eringen and Suhubi (1974) and for a recent survey article to Beatty (1987). In summary, we consider first the undeformed state of an isotropic, homogeneous, hyperelastic body and suppose that in this reference state the typical material particle is at a point with co-ordinates (X_1, X_2, X_3). When the body is subsequently deformed the co-ordinates of the site occupied by that particle at time t are (x_1, x_2, x_3), where $x_i = x_i(X_1, X_2, X_3, t)$, and Cartesian co-ordinates are used throughout. From these functions x_i we construct the deformation gradient tensor $\mathbf{F}$ by the rule

$$F_{iK} = \frac{\partial x_i}{\partial X_K}, \quad (i, K = 1, 2, 3). \tag{2}$$

By the polar decomposition theorem $\mathbf{F}$ admits a unique decomposition of the form

$$\mathbf{F} = \mathbf{VR}, \tag{3}$$

where $\mathbf{V}$ is a positive symmetric tensor and $\mathbf{R}$ is a proper orthogonal tensor. Thus

$$\mathbf{V}^{\mathrm{T}} = \mathbf{V}, \quad \mathbf{RR}^{\mathrm{T}} = \mathbf{I}, \tag{4}$$

where $\mathbf{I}$ is the identity and T denotes transpose. The eigenvalues of $\mathbf{V}$ are the principal stretches λ_1, λ_2 and λ_3. It is usually more convenient to work with λ_i^2 rather than λ_i so we construct the Cauchy deformation tensor $\mathbf{B}$ and its inverse $\mathbf{B}^{-1}$ by the rule

$$\mathbf{B} = \mathbf{FF}^{\mathrm{T}} = \mathbf{V}^2. \tag{5}$$

The invariants of $\mathbf{B}$ are denoted by I_1, I_2, I_3 where

$$I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{6}$$

For material that is compressible and free from internal constraint we denote the strain-energy per unit undeformed volume by W and then, since the principal axes of stress are coincident with those of stretch, clearly

$$\mathrm{d}W = \tau_1\lambda_2\lambda_3\,\mathrm{d}\lambda_1+\tau_2\lambda_3\lambda_1\,\mathrm{d}\lambda_2+\tau_3\lambda_1\lambda_2\,\mathrm{d}\lambda_3, \tag{7}$$

where the τ_i are the principal stresses, for all $\mathrm{d}\lambda_1$, $\mathrm{d}\lambda_2$, $\mathrm{d}\lambda_3$ without restriction. So

$$\tau_i = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_i}. \tag{8}$$

Then if W is regarded as a function of I_1, I_2, I_3 the stress tensor τ is given by

$$\tau = 2I_3^{-1/2}[W_1\mathbf{B}+(W_2I_2+W_3I_3)\mathbf{I}-W_2I_3\mathbf{B}^{-1}], \tag{9}$$

where $W_i = \partial W/\partial I_i$.

We consider now an elastic cylinder, infinite in length and circular in cross-section. Upon this cylinder we impose steady normal and axial tractions, denoting their values in the deformed state by τ_1 and τ_3 respectively. The consequent deformation, which we call the primary deformation, is given by

$$x_1 = \mu X_1, \quad x_2 = \mu X_2, \quad x_3 = \lambda X_3, \quad (\lambda, \mu \text{ constants}), \tag{10}$$

and so from (9), (10),

$$\tau_1 = 2I_3^{-1/2}[\mu^2 W_1 + (W_2 I_2 + W_3 I_3) - \mu^{-2} W_2 I_3],$$
$$\tau_3 = 2I_3^{-1/2}[\lambda^2 W_1 + (W_2 I_2 + W_3 I_3) - \lambda^{-2} W_2 I_3], \tag{11}$$

in which the derivatives of W are evaluated at the point

$$I_1 = \lambda^2 + 2\mu^2, \quad I_2 = \mu^2(2\lambda^2 + \mu^2), \quad I_3 = \lambda^2\mu^4. \tag{12}$$

When the strain-energy density function W is specified, (11) determine the values of λ, μ for given τ_1, τ_3.

We wish now to consider small-amplitude longitudinal vibrations of the cylinder and accordingly set

$$x_1 = \mu X_1 + \frac{x_1 u}{r}, \quad x_2 = \mu X_2 + \frac{x_2 u}{r}, \quad x_3 = \lambda X_3 + w, \tag{13}$$

where u and w depend upon r, z, t, with

$$r = \mu(X_1^2 + X_2^2)^{1/2}, \quad z = \lambda X_3. \tag{14}$$

The vibrations of the cylinder are regarded as perturbations of the deformed state (10) and in the subsequent calculation we retain only the first powers of u, w and their derivatives. From (2), (5), (9), (13) we calculate the stress perturbations $\hat{\tau}_{ij}$. The non-vanishing $\hat{\tau}_{ij}$ comprise the components $\hat{\tau}_{rr}$, $\hat{\tau}_{\theta\theta}$, $\hat{\tau}_{zz}$ and $\hat{\tau}_{rz}$ (where the suffixes r, θ, z indicate radial, azimuthal and axial directions). We first introduce the notation, which is closely similar to that used in Eringen and Suhubi (1974),

$$\Theta = 2I_3^{1/2}W_3, \quad \Phi = 2I_3^{-1/2}W_1, \quad \Psi = 2I_3^{-1/2}W_2,$$
$$A = 2I_3^{-1/2}W_{11}, \quad B = 2I_3^{-1/2}W_{22}, \quad C = 2I_3^{-1/2}W_{33},$$
$$D = 2I_3^{-1/2}W_{23} \quad E = 2I_3^{-1/2}W_{31}, \quad F = 2I_3^{-1/2}W_{12}, \tag{15}$$

and then find

$$\hat{\tau}_{rr} = \beta_1 u_{,r} + \frac{\beta_7 u}{r} + \beta_2 w_{,z},$$
$$\hat{\tau}_{\theta\theta} = \beta_7 u_{,r} + \frac{\beta_1 u}{r} + \beta_2 w_{,z},$$
$$\hat{\tau}_{zz} = \beta_3\left(u_{,r} + \frac{u}{r}\right) + \beta_4 w_{,z},$$
$$\hat{\tau}_{rz} = \beta_6 u_{,z} + \beta_5 w_{,r}, \tag{16}$$

where

$$\beta_1 = \Delta_1 + \mu^2\Phi + \Theta + (\lambda^2\mu^2 + \mu^4)\Psi,$$
$$\beta_2 = \Delta_2 - \mu^2\Phi + \Theta + (\lambda^2\mu^2 - \mu^4)\Psi,$$
$$\beta_3 = \Delta_2 - \lambda^2\Phi + \Theta,$$
$$\beta_4 = \Delta_3 + \lambda^2\Phi + \Theta + 2\lambda^2\mu^2\Psi,$$
$$\beta_5 = \mu^2(\Phi + \mu^2\Psi), \quad \beta_6 = \lambda^2(\Phi + \mu^2\Psi),$$
$$\beta_7 = \Delta_1 - \mu^2\Phi + \Theta + (\mu^4 - \lambda^2\mu^2)\Psi, \tag{17}$$

with

$$\Delta_1 = 2\mu^4[A+(\lambda^2+\mu^2)^2B+\lambda^4\mu^4C+2\lambda^2\mu^2(\lambda^2+\mu^2)D+2\lambda^2\mu^2E+2(\lambda^2+\mu^2)F],$$

$$\Delta_2 = 2\lambda^2\mu^2[A+2\mu^2(\lambda^2+\mu^2)B+\lambda^2\mu^6C+\mu^4(3\lambda^2+\mu^2)D+\mu^2(\lambda^2+\mu^2)E+(\lambda^2+3\mu^2)F],$$

$$\Delta_3 = 2\lambda^4[A+4\mu^4B+\mu^8C+4\mu^6D+2\mu^4E+4\mu^2F].$$

Eringen and Suhubi (1974) consider only the case in which τ_1 vanishes. In this special case our expressions (16), (17) agree with theirs. From (11), (17) we have the important results

$$\beta_2-\beta_3 = \beta_6-\beta_5 = \tau_3-\tau_1(\equiv \tau, \text{ say}). \tag{18}$$

In the absence of body forces, the equations of motion (to the present approximation) are

$$\rho\ddot{u} = \hat{\tau}_{rr,r}+\hat{\tau}_{rz,z}+\frac{(\hat{\tau}_{rr}-\hat{\tau}_{\theta\theta})}{r},$$

$$\rho\ddot{w} = \hat{\tau}_{rz,r}+\frac{\hat{\tau}_{rz}}{r}+\hat{\tau}_{zz,z}, \tag{19}$$

where ρ is the density in the deformed state, so that

$$\rho\lambda\mu^2 = \rho_0, \tag{20}$$

in which ρ_0 denotes the density in the reference state.

We consider only the case in which the traction upon the perturbed curved surface is always normal and always of magnitude τ_1, so that the boundary condition

$$\tau_{ij}n_j = \tau_1 n_i, \tag{21}$$

where the unit normal vector $\mathbf{n} = (1, 0, -u_z)$, reduces to $\hat{\tau}_{rr} = 0$, $\hat{\tau}_{rz} = \tau u_z$ on $r = a$. So from (16), (18), (19) it is readily seen that the boundary condition on the curved surface $r = a$ may be written as

$$\hat{\tau}_{rr} = 0, \quad u_z+w_r = 0 \quad \text{on } r = a. \tag{22}$$

Notice that a has been used to denote the cylinder radius after the primary deformation has been made; in the reference state the cylinder radius is $a_0 = \mu^{-1}a$.

It is possible to solve the problem summarized by (16), (19), (22) now and in particular to obtain the dispersion equation, exactly in terms of Bessel functions but the result is very complicated and extremely difficult to survey. In many applications, however, ka is small compared to unity and we exploit this fact now in order to obtain an approximate solution. Accordingly we expand the radial part of the displacement in ascending powers of kr and assume that the transverse and axial components of displacement in the lowest mode contain only odd and even powers of r respectively, as in Pochhammer's original investigation (1876), and so set

$$u = \bar{\alpha}\sum_{n=0}^{\infty} a_n(kr)^{2n+1}\exp[i(\omega t-kz)],$$

$$w = -i\bar{\alpha}\sum_{n=0}^{\infty} b_n(kr)^{2n}\exp[i(\omega t-kz)], \tag{23}$$

where $\bar{\alpha}$ is an arbitrary constant and the a_n, b_n are constants whose values are yet to be determined, we find by equating coefficients of powers of r in (19) that

$$(X-\beta_6)a_n+4(n+1)(n+2)\beta_1\alpha_{n+1}-2(n+1)(\beta_2+\beta_5)b_{n+1}=0,$$
$$(X-\beta_4)b_n+2(n+1)(\beta_3+\beta_6)a_n+4(n+1)^2\beta_5 b_{n+1}=0, \tag{24}$$

for $n = 0, 1, 2, \ldots$, where

$$X = \rho\omega^2/k^2. \tag{25}$$

The series (23) are clearly convergent for all values of the parameters. Also from (16), (22), (23)

$$\sum_{n=0}^{\infty}[a_n+2(n+1)b_{n+1}](ka)^{2n}=0,$$
$$\sum_{n=0}^{\infty}[((2n+1)\beta_1+\beta_7)a_n-\beta_2 b_n](ka)^{2n}=0. \tag{26}$$

We now write down the asymptotic developments (for small ka) of a_n, b_n, X thus

$$a_n=\sum_{m=0}^{\infty}a_n^{(m)}(ka)^{2m},\quad b_n=\sum_{m=0}^{\infty}b_n^{(m)}(ka)^{2m},\quad X=\sum_{m=0}^{\infty}X^{(m)}(ka)^{2m}. \tag{27}$$

The substitution of (27) into (24), (26) and the equating of coefficients of powers of (ka) then yields sufficient equations to enable us to find all the derived coefficients in (23), (27). Without loss of generality we may take

$$b_0^{(0)}=\frac{(\beta_1+\beta_7)}{\beta_2},\quad b_0^{(m)}=0 \text{ for } m>0. \tag{28}$$

Then we soon find

$$a_0^{(0)}=1,\quad b_1^{(0)}=-\tfrac{1}{2},\quad X^{(0)}=\beta_4-\frac{2\beta_2^2}{(\beta_1+\beta_7)},\quad a_1^{(0)}=-\frac{\Gamma}{8},$$
$$b_2^{(0)}=\frac{[(\beta_1+\beta_7)(\beta_2+\beta_5)\Gamma-2\beta_2^2]}{32\beta_5(\beta_1+\beta_7)},\quad a_0^{(1)}=\frac{[(3\beta_1+\beta_7)\Gamma-4\beta_2]}{8(\beta_1+\beta_7)},$$
$$b_1^{(1)}=-\frac{[((3\beta_1+\beta_7)\beta_5+(\beta_1+\beta_7)\beta_2)\Gamma-2\beta_2(\beta_2+2\beta_5)]}{16\beta_5(\beta_1+\beta_7)}, \tag{29}$$

where

$$\Gamma=\frac{[(\beta_1+\beta_7)(\beta_3+\beta_4)-2\beta_2^2]}{\beta_1(\beta_1+\beta_7)}, \tag{30}$$

and so, with the aid of (18),

$$X^{(1)}=\frac{-\beta_2^2}{2(\beta_1+\beta_7)^2}\left[\beta_4-\frac{2\beta_2^2}{\beta_1+\beta_7}-\tau\right]. \tag{31}$$

Equations (23), (27), (30) give a good approximation, valid for small values of (ka), for the displacement field and hence the stress distribution. From eqns (25), (27), (29), (31), we derive the approximate form of the dispersion equation

$$\frac{\rho\omega^2}{k^2} \approx \left[\beta_4 - \frac{2\beta_2^2}{\beta_1+\beta_7}\right] - \frac{\beta_2^2}{2(\beta_1+\beta_7)^2}\left[\beta_4 - \frac{2\beta_2^2}{\beta_1+\beta_7} - \tau\right](ka)^2. \tag{32}$$

For this result we now seek a simple interpretation and representation. Consider the elastic body after the primary deformation has been attained. Suppose that by a small increase $\hat{t}_{zz}$ in the axial traction, but leaving the normal traction unaltered, we impose a further, but small, deformation so that the total deformation is now given by

$$x_1 = \mu X_1 + \eta x_1, \quad x_2 = \mu X_2 + \eta x_2, \quad x_3 = \lambda X_3 + \varepsilon x_3,$$

where η, ε are small constants. The stress increments are still governed by (16) with $u = \eta r$, $w = \varepsilon z$. We introduce quantities E^*, σ^* by the relations

$$\hat{t}_{zz} = E^*\varepsilon, \quad \sigma^* = -\eta/\varepsilon. \tag{33}$$

We call E^*, σ^* respectively the modified Young's modulus and the modified Poisson's ratio; they will depend upon λ, μ and W.

From (16), (33) we see at once that

$$E^* = \beta_4 - \frac{2\beta_2\beta_3}{\beta_1+\beta_7}, \quad \sigma^* = \frac{\beta_2}{\beta_1+\beta_7}, \tag{34}$$

so from (18) and (34), eqn (32) can be written as

$$\frac{\rho\omega^2}{k^2} \approx (E^* - 2\sigma^*\tau) - \frac{\sigma^{*2}}{2}(E^* - 2\sigma^*\tau - \tau)(ka)^2. \tag{35}$$

Now E^*, σ^* are quantities which may be readily and directly measured in an experiment in which the primary deformation is perturbed by a small incremental uniaxial tension. So (35) expresses the approximate form of the dispersion relation in terms of measurable quantities.

When the steady tractions are absent, E^*, σ^* become the usual Young's modulus E and Poisson's ratio σ so in this case (35) becomes

$$\frac{\rho_0\omega^2}{k^2} \approx E\left[1 - \frac{\sigma^2}{2}(ka)^2\right], \quad (ka \ll 1), \tag{36}$$

a result first given by Pochhammer (1876).

In Fig. 1 we compare the result (36) with the results given by the exact dispersion relation, for $\sigma = 0.1, 0.4$. The graph of (c/c_0), where $c = \omega/k$ and $c_0 = (E/\rho_0)^{1/2}$, is plotted against (ka). We see that (36) affords a good approximation over the range $0 \leqslant ka < 1/2$. It is reasonable to assume that (35) is a good approximation over the same range, at least for moderate values of τ.

When there is just an axial steady traction (so that the curved surface of the cylinder is altogether free from applied traction) the values of E^*, σ^* can be found as follows. Suppose that when the cylinder has axial stretch λ, the axial principal stretch is $\tau(\lambda)$ and the transverse stretch is $\mu(\lambda)$. Then from (33)

$$E^* = \lambda\frac{d\tau}{d\lambda}, \quad \sigma^* = -\frac{\lambda}{\mu}\frac{d\mu}{d\lambda}. \tag{37}$$

When both axial and normal steady tractions are present, however, it is generally necessary to use (17), (34) and the specific form of W in order to calculate the values of E^*, σ^* before comparing with experimental results.

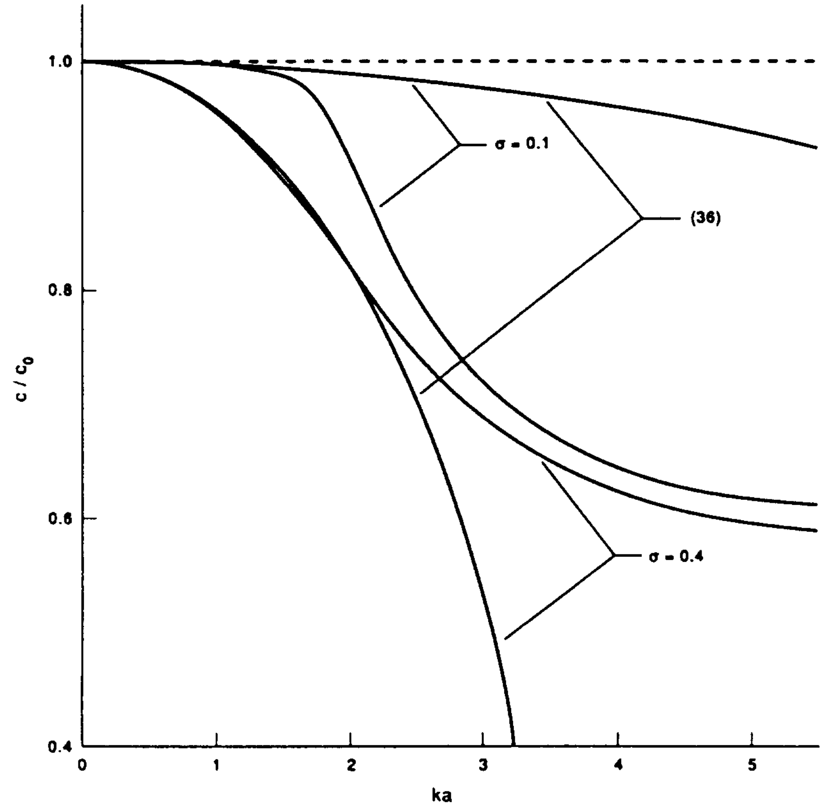

Fig. 1. The velocity diagram for longitudinal vibrations in an unstrained cylinder (Poisson's ratio $\sigma = 0.1, 0.4$). The curves show how the standardized velocity c/c_0 varies with standardized wavenumber ka. The curves marked (36) are derived from the approximation given by eqn (36); the companion curves are derived by numerical computation of the exact dispersion equation.

In order to illustrate these results we consider a generalized Blatz–Ko material [see Blatz and Ko (1962)], for which the strain-energy density function is

$$W = \bar{\mu}\left[\lambda_1\lambda_2\lambda_3 + \frac{n}{2}(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2}) + \frac{(n-1)}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2)\right], \tag{38}$$

where $\bar{\mu}, n$ are constants. The general properties of this material have been examined in detail by the authors [see Willson and Myers (1988) and Myers (1987)]. For the case of axial steady traction alone it is found from (35), with just the leading term retained, that the velocity of longitudinal waves in the lowest mode is given by

$$\left(\frac{c}{c_0}\right)^2 = \left[\frac{3n\lambda^{-2}+(n-1)\lambda^2-\frac{1}{2}n^{1/2}\lambda^2(\lambda+n-1)^{-3/2}}{4n-1-(2n)^{-1}}\right], \tag{39}$$

where $c = \omega/k$ = phase velocity of wave, $c_0 = (E_0/\rho_0)^{1/2}$ = phase velocity in undeformed state.

In Fig. 2 we show how (c/c_0) varies with λ for various selected values of n. We note that for $n < 1.059$ (approximately) there is a range of values of λ in which $(c/c_0)^2$ is negative so that very long sinusoidal waves belonging to the lowest mode cannot be propagated along the cylinder.

When both axial and normal tractions are imposed the result corresponding to (39) is too complicated to give here but in the special case $n = 1$ [this model has been used in Blatz and Ko (1962) and Ko (1963) to describe the behaviour of polyurethane rubber] we find the result

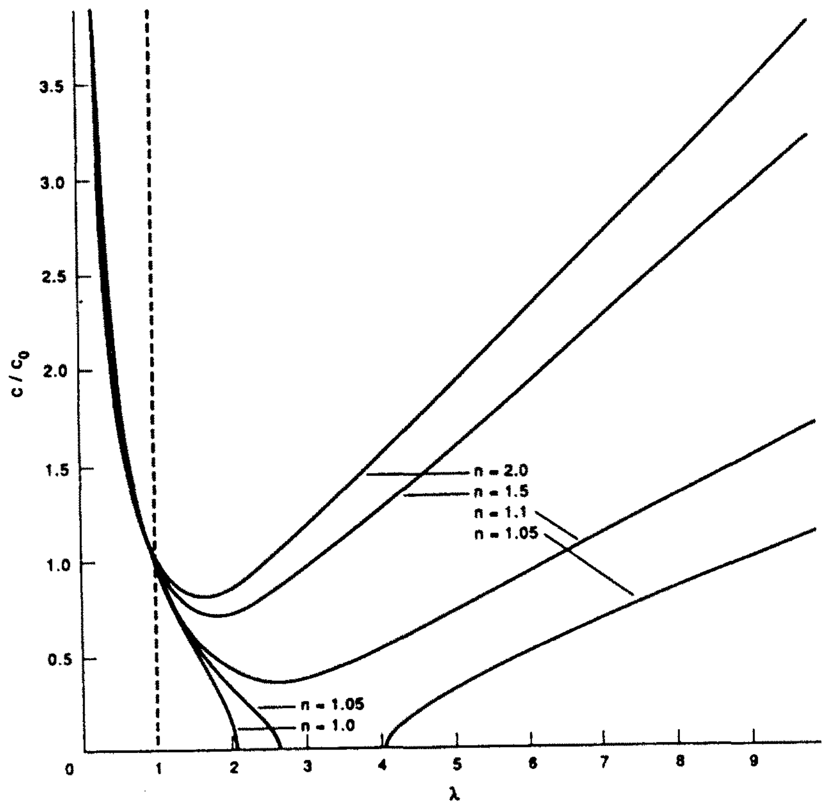

Fig. 2. The velocity diagram for longitudinal vibrations of long wavelength in a cylinder of generalized Blatz–Ko material under primary axial traction. The curves show the standardized velocity c/c_0 plotted against primary axial stretch λ for various values of the model parameter n.

$$\left(\frac{c}{c_0}\right)^2 = \frac{5+T_1-6T_3}{5(1-T_1)^{2/5}(1-T_3)^{1/5}}, \tag{40}$$

where $T_1 = \tau_1/\bar{\mu}$, $T_3 = \tau_3/\bar{\mu}$. Figure 3 shows how (c/c_0) varies with T_3 for selected values of T_1.

3. INCOMPRESSIBLE MATERIALS

An incompressible medium is an example of a material subject to an internal constraint and we now consider the modifications to the preceding analysis made necessary by this requirement. So far as possible we retain the earlier notation.

First, it is well known that for incompressible materials

$$\tau = 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1} - p\mathbf{I},$$

where p is a Lagrange multiplier representing the scalar pressure. Here W depends upon I_1 and I_2 only as $I_3 \equiv 1$. Since the steady normal traction τ_1 and axial traction τ_3 may be regarded as a hydrostatic (isotropic) pressure $-\tau_1$ together with an axial traction $\tau_3-\tau_1$ and since the hydrostatic pressure can be absorbed into the $-p\mathbf{I}$ term, it follows that for incompressible materials it is sufficient to consider only the case in which there is just an axial traction τ $(=\tau_3-\tau_1)$.

Secondly, the incompressibility condition requires directly that $I_3 = 1$ always, so in the primary deformation

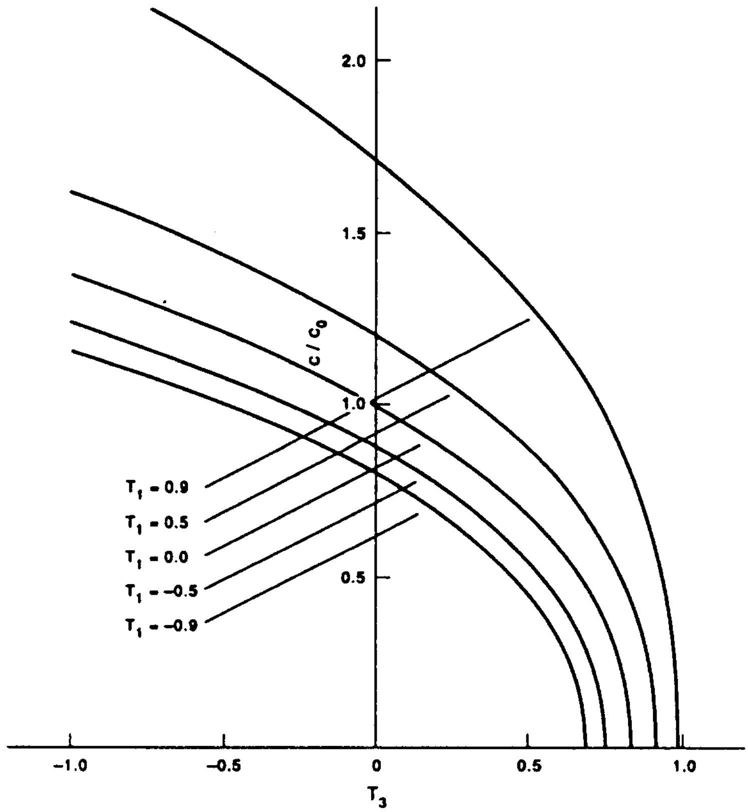

Fig. 3. The velocity diagram for longitudinal vibrations of long wavelength in a cylinder of Blatz Ko material ($n = 1$) under primary normal and axial tractions (in standardized form T_1, T_3 respectively).

$$\lambda\mu^2 = 1 \tag{41}$$

and in the perturbed deformation

$$u_{,r} + \frac{u}{r} + w_{,z} = 0. \tag{42}$$

In place of (11)

$$\tau = (\lambda^2 - \mu^2)(\Phi + \mu^2\Psi) \tag{43}$$

and for the stress-perturbations [with the aid of (42)]

$$\hat{t}_{rr} = -\hat{p} + \beta_1 u_{,r} + \beta_2 w_{,z}, \quad \hat{t}_{\theta\theta} = -\hat{p} + \frac{\beta_1 u}{r} + \beta_2 w_{,z},$$

$$\hat{t}_{zz} = -\hat{p} + \beta_4 w_{,z}, \quad \hat{t}_{rz} = \beta_6 u_{,z} + \beta_5 w_{,r}, \tag{44}$$

where $\hat{p}$ is the scalar pressure perturbation and

$$\begin{aligned} \beta_1 &= 2\mu^2\Phi + 2\lambda^2\mu^2\Psi, \\ \beta_2 &= 2(\lambda^2 - \mu^2)[\mu^2 A - \lambda^2\mu^4 B + (\mu^4 - \lambda^2\mu^2)F], \\ \beta_4 &= 2\lambda^2\Phi + 2\mu^4\Psi + 2(\lambda^2 - \mu^2)[\lambda^2 A - \mu^6 B + (\lambda^2\mu^2 - \mu^4)F], \\ \beta_5 &= \mu^2(\Phi + \mu^2\Psi), \quad \beta_6 = \lambda^2(\Phi + \mu^2\Psi). \end{aligned} \tag{45}$$

We note that

$$\bar{\beta}_6 - \bar{\beta}_5 = \tau, \tag{46}$$

so that the boundary conditions may still be written in the form (22). The calculation of the deformation and the dispersion equation can be made in the same way as in Section 2. Equation (19) stands, and we augment (23), (27) with the expansions

$$\hat{p} = k\bar{\alpha} \sum_{n=0}^{\infty} \pi_n (kr)^{2n} \exp[\mathrm{i}(\omega t - kz)], \tag{47}$$

and

$$\pi_n = \sum_{m=0}^{\infty} \pi_n^{(m)} (ka)^{2m}. \tag{48}$$

We omit the details of the calculation and give only the results. We may take $a_0 = 1$ and then

$$\begin{aligned} b_0 &= 2, \quad \pi_0^{(0)} = \bar{\beta}_1 - 2\bar{\beta}_2, \quad b_1^{(0)} = -\tfrac{1}{2}, \\ a_1^{(0)} &= -\tfrac{1}{8}, \quad X^{(0)} = \bar{\beta}_4 + \frac{\bar{\beta}_1}{2} - \bar{\beta}_2 - \tau, \end{aligned} \tag{49}$$

and then

$$\begin{aligned} \pi_1^{(0)} &= \frac{[\bar{\beta}_4 - \bar{\beta}_1/2 - 2\tau]}{2}, \quad b_2^{(0)} = \frac{X^{(0)} - \bar{\beta}_4 + 2\pi_1^{(0)} + \bar{\beta}_6}{32\bar{\beta}_5}, \\ b_1^{(1)} &= \frac{1}{16} - 2b_2^{(0)}, \quad a_1^{(1)} = \frac{1}{64} - \frac{b_2^{(0)}}{2}, \\ \pi_0^{(1)} &= -\pi_1^{(0)} - \frac{3\bar{\beta}_1}{8} + \frac{\bar{\beta}_2}{2}, \quad X^{(1)} = -\tfrac{1}{8}[X^{(0)} - \tau]. \end{aligned} \tag{50}$$

The definitions of E^*, σ^* can be retained but it is readily seen that for incompressible materials $\sigma^* = \frac{1}{2}$ always. The approximate form of the dispersion equation becomes

$$\frac{\rho_0 \omega^2}{k^2} \approx (E^* - \tau) - \frac{(E^* - 2\tau)}{8}(ka)^2, \tag{51}$$

with

$$E^* = \bar{\beta}_4 + \frac{\bar{\beta}_1}{2} - \bar{\beta}_2, \tag{52}$$

so that in terms of W we have

$$\begin{aligned} E^* - \tau &= (\lambda^2 + 2\mu^2)\Phi + 3\mu^4\Psi + 2(\lambda^2 - \mu^2)^2(A + \mu^4 B + 2\mu^2 F), \\ E^* - 2\tau &= 3\mu^2\Phi + \mu^2(4\mu^2 - \lambda^2)\Psi + 2(\lambda^2 - \mu^2)^2(A + \mu^4 B + 2\mu^2 F). \end{aligned} \tag{53}$$

For illustrative purposes we retain just the leading term in (51) so that

$$\left(\frac{c}{c_0}\right)^2 \approx \frac{(E^* - \tau)}{E_0}. \tag{54}$$

For Mooney–Rivlin material

$$W = \alpha(I_1 - 3) + \beta(I_2 - 3), \quad (\alpha, \beta \text{ constants}),$$

so that (54) becomes

$$\left(\frac{c}{c_0}\right)^2 \approx \frac{\alpha(\lambda^2 + 2\mu^2) + 3\beta\mu^4}{3(\alpha+\beta)}, \quad (\lambda\mu^2 = 1),$$

in agreement with Suhubi (1965) [see also Eringen and Suhubi (1974)]. This result is illustrated in Fig. 4 for various values of β/α.

In Fig. 5 we show how (c/c_0) varies with axial stretch λ for a model often used [see Fung (1967) and Beatty (1987)] to describe the behaviour of biological tissue

$$W = \frac{\bar{\mu}}{2\gamma}(\exp\gamma(I_1 - 3) - 1), \tag{55}$$

where $\bar{\mu}$, γ are positive constants; in this case (51), (52), (54) yield

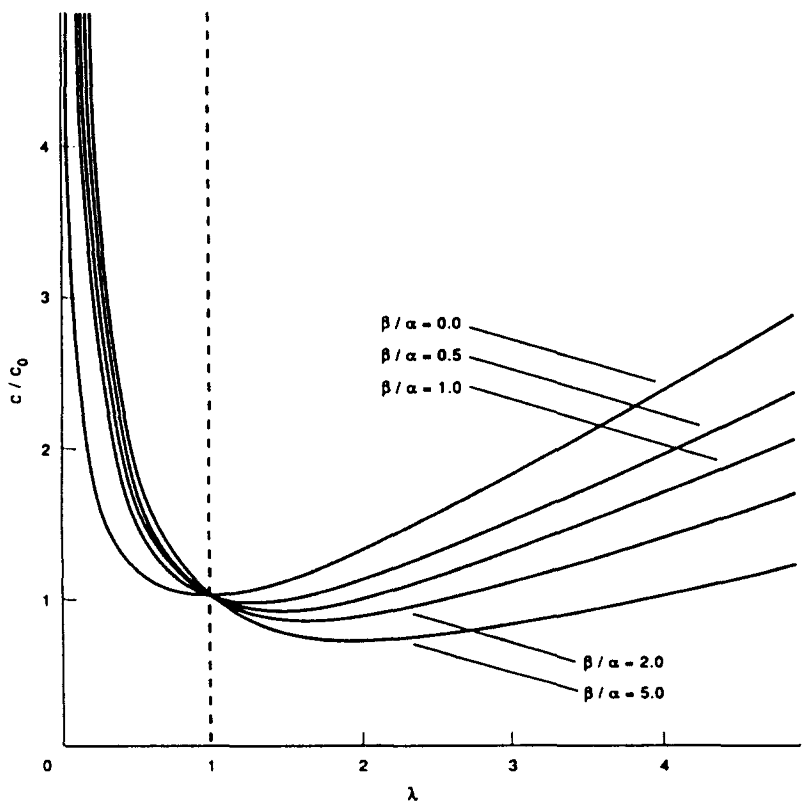

Fig. 4. The velocity diagram for longitudinal vibrations of a cylinder of Mooney–Rivlin material under primary traction. The curves show the standardized velocity c/c_0 of long waves plotted against the primary axial stretch λ for various values of the ratio β/α of the model parameters.

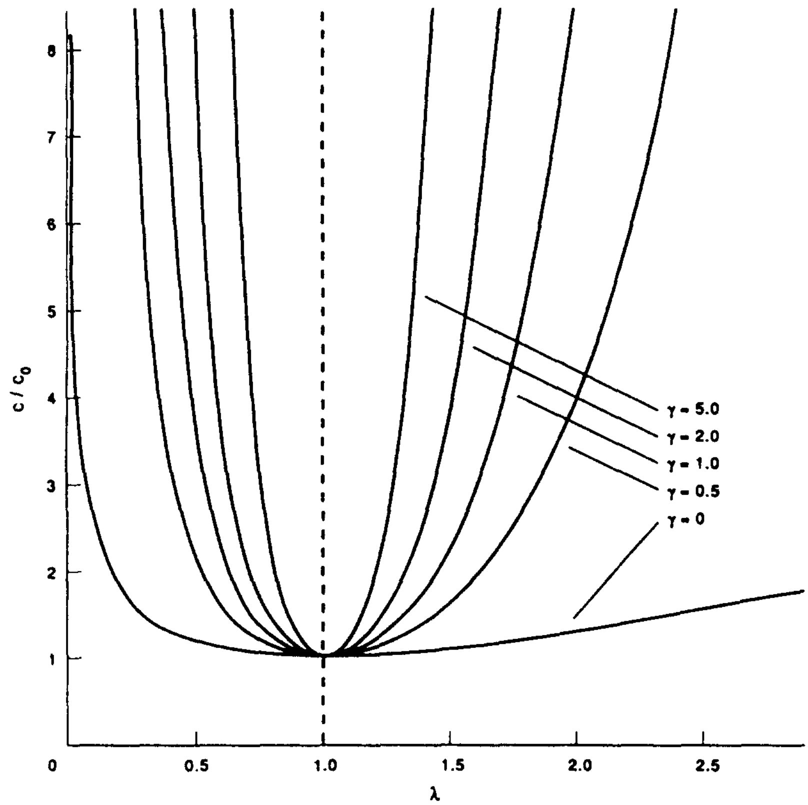

Fig. 5. The velocity diagram for longitudinal vibrations of a cylinder of biological tissue under primary traction. The curves show the standardized velocity c/c_0 of long waves plotted against the primary axial stretch λ for various values of the model parameter γ.

$$\left(\frac{c}{c_0}\right)^2 \approx \tfrac{1}{3}[\lambda^2+2\lambda^{-1}+2\gamma(\lambda^4-2\lambda+\lambda^{-2})]\exp[\gamma(\lambda^2+2\lambda^{-1}-3)]. \tag{56}$$

4. MATERIALS OBEYING THE BELL CONSTRAINT

In this section we consider the vibrations of a cylinder composed of elastic material subject to the Bell constraint [see Bell (1983)], that is,

$$\lambda_1+\lambda_2+\lambda_3 = 3. \tag{57}$$

Then with the same assumptions as before

$$\mathrm{d}W = \tau_1\lambda_2\lambda_3\,\mathrm{d}\lambda_1+\tau_2\lambda_3\lambda_1\,\mathrm{d}\lambda_2+\tau_3\lambda_1\lambda_2\,\mathrm{d}\lambda_3$$

for all $\mathrm{d}\lambda_1$, $\mathrm{d}\lambda_2$, $\mathrm{d}\lambda_3$ satisfying $\mathrm{d}\lambda_1+\mathrm{d}\lambda_2+\mathrm{d}\lambda_3 = 0$. Hence

$$\tau_i = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial\lambda_i}+\bar{K}\lambda_i, \tag{58}$$

where $\bar{K}$ is a Lagrange multiplier. So, for the Bell constraint, τ acquires an additional term $\bar{K}\mathbf{V}$, a result given in Bell (1983) but derived by a different method. It is convenient to work in terms of the invariants of $\mathbf{V}$,

$$J_1 = \lambda_1+\lambda_2+\lambda_3 = 3, \quad J_2 = \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2, \quad J_3 = \lambda_1\lambda_2\lambda_3, \tag{59}$$

and then we have at once, with $W = W(J_2, J_3)$, of course,

$$\boldsymbol{\tau} = \frac{\partial W}{\partial J_3}\mathbf{I}+K\mathbf{V}-\frac{1}{J_3}\frac{\partial W}{\partial J_2}\mathbf{V}^2, \tag{60}$$

where K is the Lagrange multiplier.† It plays a rôle analogous to the scalar pressure for incompressible materials, and must vary in space and time in just such a way as will ensure that $J_1 = 3$ always.

With the cylinder placed under normal traction τ_1 and axial traction τ_3, as before, we consider the primary deformation (10) so that

$$\lambda+2\mu = 3, \quad J_2 = 2\lambda\mu+\mu^2, \quad J_3 = \lambda\mu^2 \tag{61}$$

and

$$\tau_1 = W_3-\lambda^{-1}W_2+K\mu, \quad \tau_3 = W_3-\lambda\mu^{-2}W_2+K\lambda \tag{62}$$

since in the primary deformation

$$\mathbf{V} = \mathrm{diag}\,\{\mu, \mu, \lambda\}.$$

Throughout this section, suffixes attached to W indicate partial differentiation with respect to J_2, J_3.

Now consider the perturbed state produced by the longitudinal vibration of the cylinder, the total deformation being given by (13), (14). It is easy to show that now

$$\mathbf{V} = \begin{bmatrix} \mu(1+\mu_r) & 0 & \delta \\ 0 & \mu(1+u/r) & 0 \\ \delta & 0 & \lambda(1+w_z) \end{bmatrix}, \tag{63}$$

where

$$\delta = \frac{(\lambda^2 u_z+\mu^2 w_r)}{(\lambda+\mu)}. \tag{64}$$

Also the requirement $J_1 = 3$ yields

$$u_r+\frac{u}{r}+Lw_z = 0, \tag{65}$$

where $L = \lambda/\mu$.

A short calculation now reveals that the perturbation stresses may be written as

$$\begin{aligned} \hat{\tau}_{rr} &= \gamma_1 u_r+\gamma_2 w_z+\hat{K}\mu, \\ \hat{\tau}_{\theta\theta} &= \gamma_1 u/r+\gamma_2 w_z+\hat{K}\mu, \\ \hat{\tau}_{zz} &= \gamma_4 w_z+\hat{K}\lambda, \\ \hat{\tau}_{rz} &= \gamma_6 u_z+\gamma_5 w_r, \end{aligned} \tag{66}$$

where

† The authors understand that the result (60) has also been obtained by Beatty and Hayes and will appear shortly in the *Journal of Elasticity*.

$$\gamma_1 = K\mu - 2\lambda^{-1}W_2,$$

$$\gamma_2 = (\lambda^{-1} - \mu^{-1})W_2 + (\mu - \lambda)[(\lambda - \mu)W_{23} - W_{22} + \lambda\mu W_{33}],$$

$$\gamma_4 = K\lambda - (\lambda\mu^{-2} + \lambda^2\mu^{-3})W_2 + (\mu - \lambda)[(\lambda - \lambda^2\mu^{-1})W_{23} - \lambda^2\mu^{-2}W_{22} + \lambda\mu W_{33}],$$

$$\gamma_5 = \frac{K\mu^2}{(\lambda + \mu)} - \lambda^{-1}W_2, \quad \gamma_6 = \frac{K\lambda^2}{(\lambda + \mu)} - \lambda\mu^{-2}W_2 \tag{67}$$

and $\hat{K}$ is the perturbation in the scalar multiplier K. We see at once from (62), (67) the vital relation

$$\gamma_6 - \gamma_5 = \tau_3 - \tau_1 (\equiv \tau). \tag{68}$$

Equation (68) enables us to cast the boundary conditions in the same form as before,

$$\hat{\tau}_{rr} = 0, \quad u_z + w_r = 0 \text{ on } r = a. \tag{69}$$

The equations of motion (19) are unchanged and we may proceed to a solution as before. We augment (23), (27) with the expansions

$$\hat{K} = k\bar{\alpha} \sum_{n=0}^{\infty} \kappa_n (kr)^{2n} \exp[i(\omega t - kz)], \quad \kappa_n = \sum_{m=0}^{\infty} \kappa_n^{(m)} (ka)^{2m}. \tag{70}$$

Without loss of generality we may take $b_0 = 1$. We find

$$a_0 = \frac{L}{2}, \quad \mu\kappa_0^{(0)} = \gamma_2 - \frac{L\gamma_1}{2},$$

$$b_1^{(0)} = -\frac{L}{4}, \quad a_1^{(0)} = -\frac{L^2}{16},$$

$$X^{(0)} = \gamma_4 - L\gamma_2 + \frac{L^2\gamma_1}{2} - L\tau, \tag{71}$$

and after the next round of calculation

$$X^{(1)} = -\frac{L^2(X^{(0)} - \tau)}{8}. \tag{72}$$

By writing $u = \eta r$, $w = \varepsilon z$ in (65), (66) we find in this case

$$E^* = \gamma_4 - L\gamma_2 + \frac{L^2\gamma_1}{2}, \quad \sigma^* = \frac{L}{2}. \tag{73}$$

Hence from (71)–(73) the approximate dispersion equation can be written in the form (35), exactly as before.

It was observed in Bell (1983) that the experimental results there could be accounted for by supposing that

$$W = (4/3)^{3/4}\alpha(3 - J_2)^{3/4}, \quad (\alpha > 0 \text{ constant}). \tag{74}$$

Actually Bell (1983) contains a misprint but it is clear that (74) was intended. Accordingly we illustrate our analysis above by supposing first that W depends upon J_2 only and then specializing to the form (74).

With $W = W(J_2 \text{ only})$, we find from (67), (73) or from

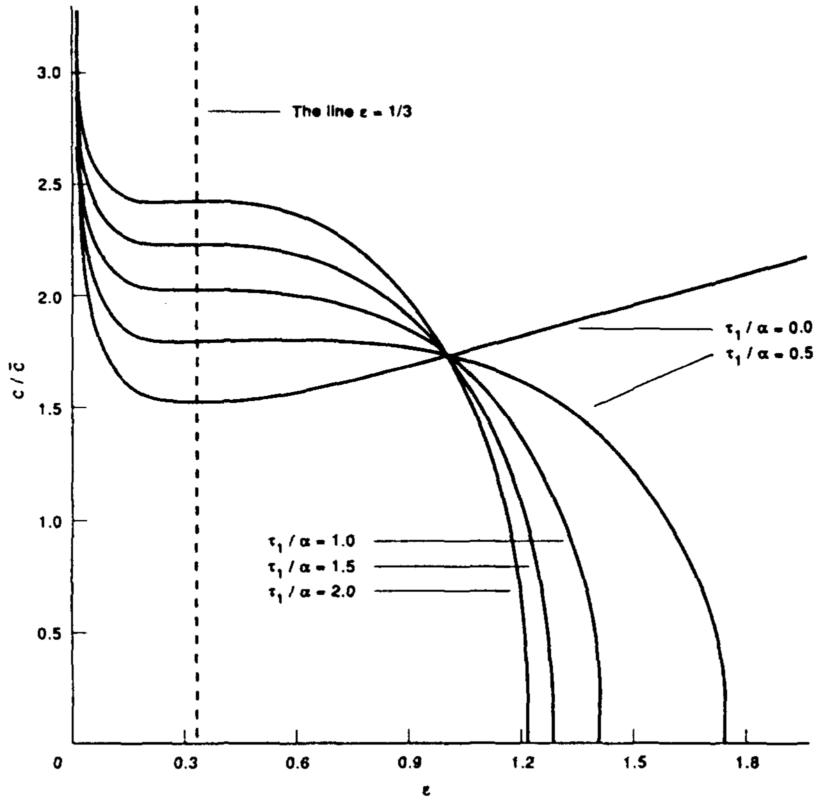

Fig. 6. The velocity diagram for longitudinal vibrations in a cylinder of material obeying the Bell constraint. The curves show the standardized velocity $c/\bar{c}$ plotted against the stretch parameter ε for specified values of the parameter τ_1/α.

$$E^* = \lambda \frac{\partial \tau_1}{\partial \lambda}(\tau_1, \lambda) \tag{75}$$

that

$$E^* = \frac{\lambda}{2\mu^3}[3\mu\tau_1 - (2\lambda+\mu)W_2 + 2\mu(\mu-\lambda)^2 W_{22}]. \tag{76}$$

If W is given by (74), then we write

$$\lambda = 1+\varepsilon, \quad \mu = 1-\varepsilon/2 \tag{77}$$

and suppose first that $\varepsilon > 0$. Then

$$E^* = \frac{3(1+\varepsilon)}{4(1-\varepsilon/2)^3}\left[(2-\varepsilon)\tau_1 + \alpha\varepsilon^{-1/2}\left(1+\frac{3\varepsilon}{2}\right)\right]. \tag{78}$$

As $\varepsilon \to 0$, $E^* \to +\infty$, so in this case for the comparison velocity we define

$$\bar{c}^2 = \alpha/\rho_0 \tag{79}$$

and then from (35), (73), (78)

$$\left(\frac{c}{\bar{c}}\right)^2 \approx \tfrac{3}{2}(1+\varepsilon)^2\left[\frac{(1-\varepsilon)\tau_1}{\alpha} + \frac{\varepsilon^{-1/2}}{2}\right]. \tag{80}$$

In Fig. 6 we show how $(c/\bar{c})$ varies with ε (> 0) for various values of τ_1/α. In particular we see that when τ_1 vanishes $(c/\bar{c})$ has a minimum at $\varepsilon = \frac{1}{3}$ and then $c/c_0 = 1.52\ldots$. Indeed, for all values of τ_1, $\varepsilon = \frac{1}{3}$ furnishes a stationary point and this is a minimum provided that $\tau_1/\alpha < 1.30\ldots$ The case $\varepsilon < 0$ may be analysed similarly.

5. SUMMARY

We have shown that in the lowest mode the longitudinal vibrations of a long circular cylinder under steady normal and axial tractions τ_1, τ_3 are governed by the approximate dispersion equation

$$\frac{\rho\omega^2}{k^2} \approx (E^* - 2\sigma^*\tau) - \frac{\sigma^{*2}}{2}(E^* - 2\sigma^*\tau - \tau)(ka)^2, \quad (ka \text{ small}),$$

where ω and k are the angular frequency and wavenumber, ρ and a the density and radius of the cylinder after the steady tractions have been imposed, and $\tau = \tau_3 - \tau_1$. The quantities E^*, σ^* are expressed in terms of the strain-energy density function W and a procedure is suggested for their measurement by experiment. When the steady tractions are absent, E^* and σ^* become E_0 and σ, the Young's modulus and Poisson's ratio for the material. The analysis is valid for cylinders composed of incompressible material or of a compressible material free from all internal constraint or of material obeying Bell's constraint. Expressions are also given for the displacement and stress fields. The results may find application in stability studies for a stressed cylinder and in the investigation of the propagation of ultrasonic pulses.

It is proposed to discuss the flexural vibrations of a loaded cylinder in a further communication.

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